# Exchange economy exercises

Consider an exchange economy with two consumers and two goods. The consumption set of each agent is $X_i = \mathbb{R}^2_+$. The total endowment of initial resources is $\omega = (1, 1)$. Determine the set of Pareto efficient allocations for each of the following pairs of preferences:

1. $u_1(x^1_1, x^1_2) = \min\{x^1_1, 2x^1_2\}, \quad u_2(x^2_1, x^2_2) = \min\{x^2_1, x^2_2\}$
2. $u_1(x^1_1, x^1_2) = x^1_1, \quad u_2(x^2_1, x^2_2) = x^2_2.$
3. $u_1(x^1_1, x^1_2) = x^1_1 x^1_2, \quad u_2(x^2_1, x^2_2) = x^2_2.$

# Solutions

1. The total initial endowment of resources is $\omega = (1,1)$. This means the sum of the allocations for each good must equal the initial endowment, i.e., $x_1^1 + x_1^2 = 1$ and $x_2^1 + x_2^2 = 1$.

   Any allocation $(x_1^1, x_2^1, x_1^2, x_2^2)$ must satisfy the constraint that the total resources are divided between the two agents. Therefore, we have:

   $$x_1^1 = x, \quad x_2^1 = y, \quad x_1^2 = 1 - x, \quad x_2^2 = 1 - y$$

   For Agent 2, their utility is maximized when $x_1^2 = x_2^2$, which means $1 - x = 1 - y$ or $x = y$. Let's see what needs to happen for Argent 1:

   If, for example, $x_1^2 > x_1^1 > x_1^1/2$ this means: $y > x > x/2$ (and therefore $1 - x > 1 - y$), then

   $$u_1(x, y) = \min\{x, 2y\} = x < y = u_1(y, y)$$

   $$u_2(1 - x, 1 - y) = \min\{1 - x, 1 - y\} = 1 - y < 1 - x = u_2(1 - y, 1 - y)$$

   That is, the allocation $(x_1^1, x_2^1, x_1^2, x_2^2) = (x, y; 1 - x, 1 - y)$ is dominated in the Pareto sense by the allocation $(x_1^1, x_2^1, x_1^2, x_2^2) = (y, y; 1 - y, 1 - y)$.

   On the other hand, if $x_2^1 < x_1^1/2 < x_1^1$ this means $y < x/2 < x$ (and therefore $1 - x < 1 - y$), then

   $$u_1(x, y) = \min\{x, 2y\} = y = u_1(y, y)$$

   $$u_2(1 - x, 1 - y) = \min\{1 - x, 1 - y\} = 1 - x < 1 - y = u_2(1 - y, 1 - y)$$

   That is, the allocation $(x_1^1, x_2^1, x_1^2, x_2^2) = (y, y; 1-y, 1-y)$ Pareto dominates the allocation $(x_1^1, x_2^1, x_1^2, x_2^2) = (x, y; 1 - x, 1 - y)$.

   Let's check for $y = x/2$, then

   $$u_1(x, x/2) = \min\{x, x\} = x$$

   $$u_2(1 - x, 1 - x/2) = \min\{1 - x, 1 - x/2\} = 1 - x$$

   therefore, as long as $\frac{x}{2} = y$ both agents are indifferent to the consumption variation of the second good.

   Let's check for $x/2 < y < x$, then

   $$u_1(x, y) = \min\{x, 2y\} = x$$

   $$u_2(1 - x, 1 - y) = \min\{1 - x, 1 - y\} = 1 - x$$

   therefore, as long as $x/2 < y < x$ both agents are indifferent to the consumption variation of the second good.

   Combining these conditions, we get the set of allocations that are Pareto efficient:

   $$\{(x_1^1, x_2^1, x_1^2, x_2^2) = (x, y; 1 - x, 1 - y) : 0 \leq x \leq 1, \frac{x}{2} \leq y \leq x\}$$

2. Since the utility function of the first agent only depends on the first good and that of the second agent only depends on the second good, the only Pareto efficient allocation is

   $$x^1 = (x_1^1, x_2^1) = (1, 0), \quad x^2 = (x_1^2, x_2^2) = (0, 1)$$

3. Agent 2 is indifferent to the first good. The only Pareto efficient allocations are of the form

   $$x^1 = (x_1^1, x_2^1) = (1, t), \quad x^2 = (x_1^2, x_2^2) = (0, 1 - t), \quad 0 \leq t \leq 1$$